# Notes of [2]

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## 1 Problem Setup

There is a *tabular episodic* MDP $\mathcal{M} = (\mathcal{S}, \mathcal{A}, \theta^*, R, H, s_1)$ where we assume the reward function $R$ is bounded within $[0, 1]$ and for simplicity we also assume $R$ is *deterministic*. In other words, only the transition probability $\mathbb{P}$ is *unknown*. We want to find a policy such that the *expected* regret incurred by this policy after $K$ episodes is minimized.

## 2 Thompson Sampling

Like *Optimism in the Face of Uncertainty*, *Thompson Sampling* dating back to [3] is another general principal guiding you how to operate in a poorly understood environment. Due to its superior empirical performance [1], it gains increasing popularity recently. Thompson Sampling is a *Bayesian* method. Basically, at the very begining, the learner equipped with this policy assumes a prior distribution $\mathcal{P}_1$ on the unknown parameter of the underlying environment i.e., $\theta^*$. At the begining of each episode $k \geq 1$, the learner just samples a virtual environment from the posterior distribution $\mathcal{P}_k$ on $\theta^*$ which is derived based on $\mathcal{P}_{k-1}$ and the history in the $(k-1)$th episode via Bayes' Theorem and then takes the optimal policy assuming the underlying model is the sampled one. The following pseudocode shows the aforementioned learning procedure.

**Algorithm 1:** Thompson Sampling

```
initialization: prior distribution P_1
for episode k = 1 to K do
    compute posterior distribution P_k = P_1 | H_k
    sample θ_k from P_k and compute the optimal policy π_k
    for step h = 1 to H do
        observe state x_{k,h}
        take action a_{k,h} = π_k(x_{k,h})
```

Denote the value function starting from time $t$ under model $M'$ using policy $\pi'$ by $V_{\pi',t}^{M'}$. Given a prior distribution $\mathcal{P}_1$ on transition probability $\theta^*$, the expected *Bayesian* regret is defined by

$$\mathcal{BR}_K^{\pi} \stackrel{\text{def}}{=} \mathbb{E}_{\theta^* \sim \mathcal{P}_1}\left[\mathbb{E}\left[\sum_{k=1}^{K}(V_{*,1}^{M^*} - V_{\pi_k,1}^{M^*})(x_{k,1}) \mid \theta^*\right]\right], \tag{1}$$

where the initial state for each episode can be either randomized or *adversarial*.

## 3 Notations and Definitions

| | |
|---|---|
| $[n]$ | $\{1, 2, \ldots, n\}$ |
| $\mathcal{A}$ | action space |
| $A$ | $\lvert\mathcal{A}\rvert$ |
| $\mathcal{S}$ | state space |
| $S$ | $\lvert\mathcal{S}\rvert$ |
| $H$ | horizon |
| $K$ | # of episodes |
| $T$ | $HK$ |
| $R : \mathcal{S} \times \mathcal{A} \to [0, 1]$ | *known* reward function |
| $\theta^* : \mathcal{S} \times \mathcal{A} \to \Delta(\mathcal{S})$ | transition probability of the underlying MDP |
| $\pi = (\pi_1, \ldots, \pi_K)$ | an arbitrary policy where $\pi_k$ is the policy in the $k$th episode |
| $V_{\pi',t}^{M'}$ | value function starting from time $t$ under model $M'$ using policy $\pi'$ |
| $x_{k,1}$ | initial state of the $k$th episode |
| $(x_{k,h}, a_{k,h})$ | state-action pair in the $k$th episode and at the $h$th time step |
| $\mathcal{H}_k$ | history before the $k$th episode $(x_{1,1}, a_{1,1}, \ldots, x_{1,H+1}, \ldots, x_{k-1,1}, a_{k-1,1} \ldots, x_{k-1,H+1})$ |
| $\mathcal{M}_k$ | sampled virtual model with transition probability $\theta_k$ right before the $k$-th episode |
| $N_k(x, a)$ | number of hits of state-action pair $(x, a)$ *before* the $k$th episode |
| $\rho$ | an arbitrary transition probability |
| $V$ | an arbitrary value function |
| $(\rho V)(x, a)$ | $\sum_{y \in \mathcal{S}} \rho(y \mid x, a) V(y)$ |
| $\mathcal{BR}_K^{\pi}$ | Bayesian regret incurred by policy $\pi$ |
| $\mathcal{P}_k$ | posterior distribution right *before* the $k$th episode |

## 4 Theorem

In this lecture, we are going to show

**Theorem 1.** *When $T \geq \sqrt{SA}$, the expected Bayesian regret i.e., (1) of Algorithm 1 is bouned by $\widetilde{\mathcal{O}}(HS\sqrt{AT})$.*

**Remark 2.** *The theorem holds for any prior distribution.*

*Proof.* In the subsequent part, unless otherwise specified, the expectation operator is taken over all random variables. Note that $\theta^*$ is treated as a random variable. Rewrite $\mathcal{BR}_K^\pi$ we have

$$\begin{aligned}
(1) &= \sum_{k=1}^K \mathbb{E}\left[(V_{*,1}^{M^*} - V_{\pi_k,1}^{M^*})(x_{k,1})\right] \\
&= \sum_{k=1}^K \left(\mathbb{E}\left[(V_{*,1}^{M^*} - V_{\pi_k,1}^{M_k})(x_{k,1})\right] + \mathbb{E}\left[(V_{\pi_k,1}^{M_k} - V_{\pi_k,1}^{M^*})(x_{k,1})\right]\right) \\
&= \sum_{k=1}^K \mathbb{E}\left[(V_{*,1}^{M^*} - V_{\pi_k,1}^{M_k})(x_{k,1})\right] + \sum_{k=1}^K \mathbb{E}[\widetilde{\Delta}_k], \qquad (2)
\end{aligned}$$

where we have defined $\widetilde{\Delta}_k \overset{\text{def}}{=} (V_{\pi_k,1}^{M_k} - V_{\pi_k,1}^{M^*})(x_{k,1})$.

**Lemma 3.** $\mathbb{E}[V_{*,1}^{M^*}(x_{k,1})] = \mathbb{E}[V_{\pi_k,1}^{M_k}(x_{k,1})]$.

*Proof.* Just note that

$$\begin{aligned}
\mathbb{E}[V_{*,1}^{M^*}(x_{k,1})] &= \mathbb{E}[\mathbb{E}[V_{*,1}^{M^*}(x_{k,1}) \mid \mathcal{H}_k]] \\
&= \mathbb{E}[\mathbb{E}[V_{\pi_k,1}^{M_k}(x_{k,1}) \mid \mathcal{H}_k]] \\
&= \mathbb{E}[V_{\pi_k,1}^{M_k}(x_{k,1})].
\end{aligned}$$

□

Applying Lemma 3 in (2), we obtain

$$(1) = \sum_{k=1}^K \mathbb{E}[\widetilde{\Delta}_k].$$

Next we focus on bounding $\widetilde{\Delta}_k$. Note that

$$\begin{aligned}
\mathbb{E}[\widetilde{\Delta}_k \mid M^*, M_k] &= \mathbb{E}[(V_{\pi_k,1}^{M_k} - V_{\pi_k,1}^{M^*})(x_{k,1}) \mid M^*, M_k] \\
&= \mathbb{E}[(\theta_k V_{\pi_k,2}^{M_k})(x_{k,1}, a_{k,1}) - (\theta^* V_{\pi_k,2}^{M^*})(x_{k,1}, a_{k,1}) \mid M^*, M_k] \\
&= \mathbb{E}[((\theta_k - \theta^*) V_{\pi_k,2}^{M_k})(x_{k,1}, a_{k,1}) + (V_{\pi_k,2}^{M_k} - V_{\pi_k,2}^{M^*})(x_{k,2}) \mid M^*, M_k] \\
&\quad + \mathbb{E}[(\theta^* (V_{\pi_k,2}^{M_k} - V_{\pi_k,2}^{M^*}))(x_{k,1}, a_{k,1}) - (V_{\pi_k,2}^{M_k} - V_{\pi_k,2}^{M^*})(x_{k,2}) \mid M^*, M_k] \\
&= \mathbb{E}[((\theta_k - \theta^*) V_{\pi_k,2}^{M_k})(x_{k,1}, a_{k,1}) + (V_{\pi_k,2}^{M_k} - V_{\pi_k,2}^{M^*})(x_{k,2}) \mid M^*, M_k],
\end{aligned}$$

where in the second last inequality we have used

$$\mathbb{E}[(\theta^*(V^{M_k}_{\pi_k,2} - V^{M^*}_{\pi_k,2}))(x_{k,1}, a_{k,1}) - (V^{M_k}_{\pi_k,2} - V^{M^*}_{\pi_k,2})(x_{k,2}) \mid M^*, M_k] = 0.$$

By recursion and law of total expectation, we derive

$$\begin{aligned}
\mathbb{E}[\widetilde{\Delta}_k] &= \sum_{t=1}^{H} \mathbb{E}[((\theta_k - \theta^*)V^{M_k}_{\pi_k,t+1})(x_{k,t}, a_{k,t})] \\
&\le \sum_{t=1}^{H} \mathbb{E}\left[\|(\theta_k - \theta^*)(x_{k,t}, a_{k,t})\|_1 \cdot \left\|V^{M_k}_{\pi_k,t+1}\right\|_\infty\right] \\
&\le H \cdot \sum_{t=1}^{H} \mathbb{E}[\|(\theta_k - \theta^*)(x_{k,t}, a_{k,t})\|_1], \qquad (3)
\end{aligned}$$

where in the first inequality we have used Hölder's inequality and in the last inequality we apply $\left\|V^{M_k}_{\pi_k,t+1}\right\|_\infty \le H$.

Let $\bar{\theta}_k(\cdot \mid s, a)$ be the empirical transition probability before the $k$th episode. We define $\mathcal{M}_k$ as the set of models such that its transition probability $\theta$ satisfies $|\bar{\theta}_k(\cdot \mid s, a) - \theta(\cdot \mid s, a)| \le C\sqrt{\frac{S\ln(SAT)}{1 \vee N_k(s,a)}}$ for all $(s, a) \in \mathcal{S} \times \mathcal{A}$ where $C$ is a universal constant which will be defined later. According to Theorem 4, we know that there exists a constant $C > 0$ such that $\mathbf{Pr}(M_k \notin \mathcal{M}_k) \le 1/K$ and $\mathbf{Pr}(M^* \notin \mathcal{M}_k) \le 1/K$. Hence

$$\begin{aligned}
(1) &= \sum_{k=1}^{K} \mathbb{E}[\widetilde{\Delta}_k] \\
&\le \sum_{k=1}^{K} \mathbb{E}[\widetilde{\Delta}_k \mathbb{1}(M_k \in \mathcal{M}_k, M^* \in \mathcal{M}_k)] + H \cdot \sum_{k=1}^{K} (\mathbf{Pr}(M_k \notin \mathcal{M}_k) + \mathbf{Pr}(M^* \in \mathcal{M}_k)),
\end{aligned}$$

according to $\widetilde{\Delta}_k \le H$ and a union bound. Recall $\mathbf{Pr}(M_k \notin \mathcal{M}_k) \le 1/K$ and $\mathbf{Pr}(M^* \notin \mathcal{M}_k) \le 1/K$, we further obtain

$$\begin{aligned}
(1) &\lesssim \sum_{k=1}^{K} \mathbb{E}[\widetilde{\Delta}_k \mathbb{1}(M_k \in \mathcal{M}_k, M^* \in \mathcal{M}_k)] \\
&= \sum_{k=1}^{K} \mathbb{E}[\widetilde{\Delta}_k \mathbb{1}(M_k \in \mathcal{M}_k, M^* \in \mathcal{M}_k)] \qquad (4)
\end{aligned}$$

Putting (3) back into (4), we have

$$\begin{aligned}
(1) &\lesssim \sum_{k=1}^{K} \mathbb{E}\left[\sum_{t=1}^{H} ((\theta_k - \theta^*)V^{M_k}_{\pi_k,t+1})(x_{k,t}, a_{k,t}) \cdot \mathbb{1}(M_k \in \mathcal{M}_k, M^* \in \mathcal{M}_k)\right] \\
&\le \sum_{k=1}^{K} \mathbb{E}\left[\sum_{h=1}^{H} CH\sqrt{\frac{S\ln(SAT)}{1 \vee N_k(x_{k,h}, a_{k,h})}} \cdot \mathbb{1}(M_k \in \mathcal{M}_k, M^* \in \mathcal{M}_k)\right] \\
&= \mathbb{E}\left[CH \cdot \sum_{k=1}^{K}\sum_{h=1}^{H} \sqrt{\frac{S\ln(SAT)}{1 \vee N_k(x_{k,h}, a_{k,h})}} \cdot \mathbb{1}(M_k \in \mathcal{M}_k, M^* \in \mathcal{M}_k)\right], \qquad (5)
\end{aligned}$$

where in the second last inequality we have used $|\bar{\theta}_k(\cdot \mid s,a) - \theta_k(\cdot \mid s,a)| \le C\sqrt{\frac{S\ln(SAT)}{1 \vee N_k(s,a)}}$ and $|\bar{\theta}_k(\cdot \mid s,a) - \theta^*(\cdot \mid s,a)| \le C\sqrt{\frac{S\ln(SAT)}{1 \vee N_k(s,a)}}$ when $M_k \in \mathcal{M}_k$ and $M^* \in \mathcal{M}_k$.

Note that

$$
\begin{aligned}
\sum_{k=1}^{K}\sum_{h=1}^{H}\sqrt{\frac{S\ln(SAT)}{1 \vee N_k(x_{k,h},a_{k,h})}} &\lesssim \sqrt{S\ln(SAT)} \cdot \sum_{(s,a)\in\mathcal{S}\times\mathcal{A}} \sum_{t=0}^{N_K(s,a)} \sqrt{\frac{1}{1 \vee t}} \\
&\le \sqrt{S\ln(SAT)} \cdot \left( \sum_{(s,a)\in\mathcal{S}\times\mathcal{A}} 2\sqrt{N_K(s,a)} + SA \right) \\
&\le 2S\sqrt{AT\ln(SAT)} = \widetilde{\mathcal{O}}(S\sqrt{AT}), \qquad (6)
\end{aligned}
$$

where in the third last inequality we have used $\sum_{(s,a)\in\mathcal{S}\times\mathcal{A}} \sqrt{N_K(s,a)} \le \sqrt{SAT}$ which is due to Cauchy-Schwarz inequality and $T \ge \sqrt{SA}$.

Putting (6) back into (5), we prove this theorem. □

# 5 Tools

**Theorem 4** ([4]). *Let $P$ be a probability distribution on the set $\mathcal{S} = \{1, \ldots, S\}$. Let $X_1$, $X_2$, ..., $X_m$ be i.i.d. random variables distributed according to $P$. Then, for all $\epsilon > 0$, it holds that*

$$\mathbf{Pr}(\left\|P - \bar{P}\right\|_1 \ge \epsilon) \le (2^S - 2)\exp(-m\epsilon^2/2),$$

*where $\bar{P}$ is the empirical estimation of $P$ defined as $\bar{P}(i) = \frac{\sum_{j=1}^{m} \mathbb{1}(X_j = i)}{m}$.*

# References


[1] Olivier Chapelle and Lihong Li. An empirical evaluation of thompson sampling. In *NIPS*, pages 2249–2257, 2011.

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[3] William R Thompson. On the likelihood that one unknown probability exceeds another in view of the evidence of two samples. *Biometrika*, 25(3/4):285–294, 1933.

[4] Tsachy Weissman, Erik Ordentlich, Gadiel Seroussi, Sergio Verdu, and Marcelo J Weinberger. Inequalities for the l1 deviation of the empirical distribution. *Hewlett-Packard Labs, Tech. Rep*, 2003.