

Notes of [2]

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1 Problem Setup

There is a *tabular infinite undiscounted* MDP $M^* = (\mathcal{S}, \mathcal{A}, \theta^*, c, s_1)$ where cost function c is bounded within $[0, 1]$ and s_1 is the initial state which can be either randomized or *adversarial*. For simplicity, we assume the cost function c is *deterministic* and known beforehand. In other words, only the transition probability θ^* is *unknown*. We want to find a policy such that the expected cost incurred by this policy after T time steps is minimized.

In infinite undiscounted MDP, the expected average cost per step for any policy π is defined as

$$\ell_\pi \stackrel{\text{def}}{=} \limsup_{T \rightarrow +\infty} \frac{1}{T} \cdot \mathbb{E} \left[\sum_{t=1}^T c(x_t, a_t) \right],$$

where x_t and a_t denotes the state and action pair at the t th time step. Note that we have removed the dependency on policy to simplify the notations. Let π^* be the optimal policy such that $\ell_{\pi^*} = \min_{\pi'} \ell_{\pi'}$. And the *frequentist* regret is defined by

$$\mathcal{R}_T^\pi \stackrel{\text{def}}{=} \sum_{t=1}^T c(x_t, a_t) - T\ell_{\pi^*}.$$

1.1 Weakly Communicating MDP

To make it possible to suffer a sub-linear regret, we also need to make some restrictions on the underlying MDP. Here, we assume the underlying MDP is *weakly communicating*.

Definition 1. An MDP is *weakly communicating* iff the state space \mathcal{S} can be decomposed into two parts \mathcal{S}_1 and \mathcal{S}_2 such that every state in \mathcal{S}_1 is reachable from other states in \mathcal{S}_1 under some policy, whereas all states in \mathcal{S}_2 are transient under all policies.

Remark 1. A state $s \in \mathcal{S}$ is said *transient* if upon entering the state, the MDP will never return back to it in the future. More precisely, let $s_1 = s$ and $T_s = \inf_t \{t \geq 1 : x_t = s\}$ be the first time the MDP returns back to s . It holds that $\Pr(T_s = +\infty) = 1$.

The intuition to introduce such a concept is to avoid trap states. For example we can construct an MDP as the following (see Figure 1):

- i. $\mathcal{S} = \{s_1, s_2\}$

- ii. $\mathcal{A} = \{a_1, a_2\}$
- iii. $\theta^*(s_1 | s_1, a_1) = 1, \theta^*(s_2 | s_1, a_2) = 1, \theta^*(s_2 | s_2, a_1) = 1, \theta^*(s_2 | s_2, a_2) = 1$
- iv. $c(s_1, a_1) = 0.5, c(s_1, a_2) = 1, c(s_2, a_1) = 1, c(s_2, a_2) = 1$
- v. $x_1 = s_1$

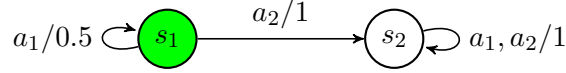


Figure 1: A counterexample

It is clear that the optimal policy is $\pi^* = a_1$. However, no policy could achieve $o(T)$. We prove this by contradiction. Suppose such a policy exists. We call it π' . Then during the first $T/2$ steps, it must not try action a_2 . Otherwise, the regret would be at least $0.25T = \Omega(T)$. A key observation is that when we change the transition probability to the following case,

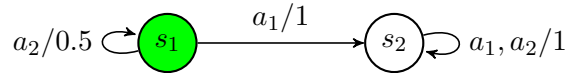


Figure 2: Counterexample with Changed Transition Probability

π' will not change its behavior in the first $T/2$ steps since it does not even try action a_2 . Note that during the second $T/2$ steps, it will get stuck in state s_2 and this incurs a $\Omega(T)$ regret. A contradiction happens.

1.2 Optimality

Theorem 2. *There always exists a stationary deterministic policy π^* achieving the optimal expected average cost and its expected average cost satisfies*

$$\ell_{\pi^*}^{M^*} + v(x, \theta^*) = \min_{a \in \mathcal{A}} \left\{ c(x, a) + \sum_{x' \in \mathcal{S}} \theta^*(x' | x, a) v(x', \theta^*) \right\},$$

where $v(\cdot, \theta^*)$ is called the bias vector of MDP M^* .

It is easy to see if $v(\cdot, \theta^*)$ is a bias vector of model M^* , so does $v(\cdot, \theta^*) - C$ where C is an arbitrary constant. Hence w.l.o.g., we assume $\min_{x \in \mathcal{S}} v(x, \theta^*) = 0$. We also assume $\max_{x \in \mathcal{S}} v(x, \theta^*) \leq D'$.

Remark 3. *We only assume the existence of D' . We do not assume D' is known beforehand.*

From now on, we assume the underlying unknown MDP is weakly communicating and its bias vector is upper bounded by D' and only need to consider stationary deterministic policies.

2 Thompson Sampling

Like *Optimism in the Face of Uncertainty*, *Thompson Sampling* dating back to [3] is another general principle guiding you how to operate in a poorly understood environment. Due to its superior empirical performance [1], it gains increasing popularity recently.

Thompson Sampling is a *Bayesian* method. Basically, at the very beginning, the learner equipped with this policy assumes a prior distribution \mathcal{P}_1 on the unknown parameter of the underlying environment i.e., θ^* . At the beginning of each episode $k \geq 1$, the learner just samples a virtual environment from the posterior distribution \mathcal{P}_k on θ^* which is derived based on \mathcal{P}_{k-1} and the history in the $(k-1)$ th episode via Bayes' Theorem and then takes the optimal policy assuming the underlying model is the sampled one.

To apply Thompson Sampling, we need to design a stopping criteria for each episode. Before describing the stopping criteria, we introduce several notations. Let t_k and T_k denote the start time and the length of the k th episode respectively. Also let $N_t(x, a)$ be the number of visits of state-action pairs before time step t .

In the algorithm we are going to talk about, episode k finishes if one of the following situation happens:

- i. $t - t_k > T_{k-1}$ or
- ii. $\exists(x, a) \in \mathcal{S} \times \mathcal{A}$, s.t., $N_t(x, a) > 2N_{t_k}(x, a)$.

The details are described in the following Algorithm 1.

Algorithm 1: Thompson Sampling

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1 initialization: prior distribution  $\mathcal{P}_1$ , start of episode  $k = 1$  and start time  $t = 1$ 
2 while  $t \leq T$  do
3    $t_k = t$  // start time of  $k$ th episode
4   compute posterior distribution  $\mathcal{P}_k = \mathcal{P}_1 \mid \mathcal{H}_{t_k}$ 
5   sample  $\theta_k$  from  $\mathcal{P}_k$  and compute the optimal policy  $\pi_k$ 
6   while  $t \leq T$  and  $t - t_k \leq T_{k-1}$  and  $N_t(x, a) \leq 2N_{t_k}(x, a) \forall (x, a) \in \mathcal{S} \times \mathcal{A}$  do
7     observe state  $x_t$  and take action  $a_t$  according to policy  $\pi_k$ 
8      $t = t + 1$ 
9    $k = k + 1$ 

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Given a prior distribution \mathcal{P}_1 on transition probability θ^* , the expected *Bayesian* regret is defined by

$$\mathcal{BR}_T^\pi \stackrel{\text{def}}{=} \mathbb{E}_{\theta^* \sim \mathcal{P}_1} [\mathbb{E} [\mathcal{R}_T^\pi \mid \theta^*]]. \quad (1)$$

3 Notations and Definitions

$[n]$	$\{1, 2, \dots, n\}$
\mathcal{A}	action space
A	$ \mathcal{A} $
\mathcal{S}	state space
S	$ \mathcal{S} $
T	horizon of the MDP
$c : \mathcal{S} \times \mathcal{A} \rightarrow [0, 1]$	<i>known</i> cost function
$\theta^* : \mathcal{S} \times \mathcal{A} \rightarrow \Delta(\mathcal{S})$	transition probability of the underlying MDP
π_k	policy in the k th episode
x_1	initial state
(x_t, a_t)	state-action pair at the t th time step
\mathcal{H}_t	history before the t th time step $(x_1, a_1, \dots, x_{t-1}, a_{t-1}, x_t)$
$N_t(x, a)$	number of hits of state-action pair (x, a) <i>before</i> the t th time step
t_k	start time of the k th episode
T_k	length of the k th episode
\mathcal{P}_k	posterior distribution right <i>before</i> the k th episode
\mathcal{BR}_T^π	Bayesian regret incurred by policy π

4 Theorem

In this lecture, we are going to show

Theorem 4. *The expected Bayesian regret i.e., (1) incurred by Algorithm 1 is bounded by $\tilde{O}(D'S\sqrt{AT})$.*

Remark 5. *The theorem holds for any prior distribution.*

Proof. In the subsequent part, unless otherwise specified, the expectation operator is taken over all random variables. Note that θ^* is treated as a random variable. Let K be the random variable denoting the total number of episodes. W.o.l.g., we assume $t_{K+1} = T + 1$. Rewrite \mathcal{BR}_T^π we have

$$\begin{aligned}
(1) &= \mathbb{E} \left[\sum_{k=1}^K \sum_{t=t_k}^{t_{k+1}-1} c(x_t, a_t) \right] - T \cdot \mathbb{E}[\ell_{\pi^*}^{M^*}] \\
&= \mathbb{E} \left[\sum_{k=1}^K \sum_{t=t_k}^{t_{k+1}-1} \left[\ell_{\pi_k}^{M_k} + v(x_t, \theta_k) - \sum_{x' \in \mathcal{S}} \theta_k(x' | x_t, a_t) v(x', \theta_k) \right] \right] - T \cdot \mathbb{E}[\ell_{\pi^*}^{M^*}] \\
&= \underbrace{\mathbb{E} \left[\sum_{k=1}^K \sum_{t=t_k}^{t_{k+1}-1} \ell_{\pi_k}^{M_k} \right]}_{(I)} - T \cdot \mathbb{E}[\ell_{\pi^*}^{M^*}] + \underbrace{\mathbb{E} \left[\sum_{k=1}^K \sum_{t=t_k}^{t_{k+1}-1} (v(x_t, \theta_k) - v(x_{t+1}, \theta_k)) \right]}_{(II)} \\
&\quad + \underbrace{\mathbb{E} \left[\sum_{k=1}^K \sum_{t=t_k}^{t_{k+1}-1} \left[v(x_{t+1}, \theta_k) - \sum_{x' \in \mathcal{S}} \theta_k(x' | x_t, a_t) v(x', \theta_k) \right] \right]}_{(III)}, \tag{2}
\end{aligned}$$

where in the second last equality we have applied Theorem 2.

In the following part, we will try to bound (I), (II) and (III) separately.

Lemma 6. $(I) \leq \mathbb{E}[K]$.

Proof. First we note that

$$\begin{aligned}
(I) &= \mathbb{E} \left[\sum_{k=1}^K T_k \ell_{\pi_k}^{M_k} \right] - T \cdot \mathbb{E}[\ell_{\pi^*}^{M^*}] \\
&= \mathbb{E} \left[\sum_{k=1}^K \mathbb{E}[T_k \ell_{\pi_k}^{M_k} | \mathcal{H}_{t_k}] \right] - T \cdot \mathbb{E}[\ell_{\pi^*}^{M^*}] \\
&\leq \mathbb{E} \left[\sum_{k=1}^K \mathbb{E}[(T_{k-1} + 1) \ell_{\pi_k}^{M_k} | \mathcal{H}_{t_k}] \right] - T \cdot \mathbb{E}[\ell_{\pi^*}^{M^*}],
\end{aligned}$$

where in the last inequality we have used $T_k \leq T_{k-1} + 1$ which is enforced by the algorithm. Since conditioned on \mathcal{H}_{t_k} , T_{k-1} is a constant, we have $\mathbb{E}[(T_{k-1} + 1) \ell_{\pi_k}^{M_k} | \mathcal{H}_{t_k}] = (T_{k-1} + 1) \mathbb{E}[\ell_{\pi_k}^{M_k} | \mathcal{H}_{t_k}]$.

Further utilizing the relation that $\theta_k | \mathcal{H}_{t_k} = \theta^* | \mathcal{H}_{t_k}$, we derive

$$\begin{aligned}
(I) &\leq \mathbb{E} \left[\sum_{k=1}^K (T_{k-1} + 1) \mathbb{E}[\ell_{\pi^*}^{M^*} | \mathcal{H}_{t_k}] \right] - T \cdot \mathbb{E}[\ell_{\pi^*}^{M^*}] \\
&= \mathbb{E} \left[\sum_{k=1}^K (T_{k-1} + 1) \ell_{\pi^*}^{M^*} \right] - T \cdot \mathbb{E}[\ell_{\pi^*}^{M^*}] \\
&\leq \mathbb{E}[K \ell_{\pi^*}^{M^*}] \leq \mathbb{E}[K],
\end{aligned}$$

where the last inequality is due to $\ell_{\pi^*}^{M^*} \leq 1$. □

Lemma 7. $(II) \leq D' \mathbb{E}[K]$.

Proof. Just note that

$$\begin{aligned}
(II) &= \mathbb{E} \left[\sum_{k=1}^K \sum_{t=t_k}^{t_{k+1}-1} (v(x_t, \theta_k) - v(x_{t+1}, \theta_k)) \right] \\
&= \mathbb{E} \left[\sum_{k=1}^K (v(x_{t_k}, \theta_k) - v(x_{t_{k+1}}, \theta_k)) \right] \\
&\leq D' \mathbb{E}[K],
\end{aligned}$$

where in the last inequality is due to $0 \leq v(\cdot, \theta_k) \leq D'$. □

Putting Lemma 6 and Lemma 7 together, we get $(I) + (II) \leq (D' + 1) \mathbb{E}[K]$. We next take care of $\mathbb{E}[K]$ and try to give an upper bound of the expected number of episodes.

Lemma 8. $\mathbb{E}[K] = \mathcal{O}(\sqrt{SAT \ln(T)})$.

Proof. According to the stopping condition, we divide K episodes into M meta episodes such that within meta episode \tilde{e}_m , except for the last episode, all the other episodes end due to the first condition i.e., $t - t_k > T_{k-1}$, which means $T_k = T_{k-1} + 1$. Let τ_m be the start episode of meta episode τ_m . And we set $\tau_{M+1} = K + 1$.

Hence for any meta episode \tilde{e}_m , the total number of time steps $\sum_{k=\tau_m}^{\tau_{m+1}-1} T_k$ satisfies $\sum_{k=\tau_m}^{\tau_{m+1}-1} T_k \geq \sum_{k=\tau_m}^{\tau_{m+1}-2} (T_{\tau_m} + k - \tau_m) = (\tau_{m+1} - \tau_m - 1)(2T_{\tau_m} + \tau_{m+1} - \tau_m - 2)/2$. Since $T_{\tau_m} \geq 1$, we further derive $\tau_{m+1} - \tau_m \leq 1 + \sqrt{2 \sum_{k=\tau_m}^{\tau_{m+1}-1} T_k} \leq 2\sqrt{2 \sum_{k=\tau_m}^{\tau_{m+1}-1} T_k}$. Next by Cauchy-Schwarz inequality, we get

$$K = \tau_{M+1} - 1 = \sum_{m=1}^M (\tau_{m+1} - \tau_m) \leq \sum_{m=1}^M 2\sqrt{2 \sum_{k=\tau_m}^{\tau_{m+1}-1} T_k} \leq \sqrt{8MT}.$$

Note that M is at most the total number of episodes which ends due to visit number of state-action pair doubles. Hence $M = \mathcal{O}(SA \ln T)$. Using this inequality, we prove $K = \mathcal{O}(\sqrt{SAT \ln T})$ and finishes the proof of this lemma. □

In the remaining part of the proof, we focus on bounding (III). Expand $v(x_{t+1}, \theta_k)$ we derive

$$(III) = \mathbb{E} \left[\sum_{k=1}^K \sum_{t=t_k}^{t_{k+1}-1} \left[\sum_{x' \in \mathcal{S}} \theta^*(x' | x_t, a_t) v(x', \theta_k) - \sum_{x' \in \mathcal{S}} \theta_k(x' | x_t, a_t) v(x', \theta_k) \right] \right].$$

Since $v(\cdot, \cdot) \leq D'$, further by Hölder's inequality, we have

$$(III) \leq D' \mathbb{E} \left[\sum_{k=1}^K \sum_{t=t_k}^{t_{k+1}-1} \|\theta^*(\cdot | x_t, a_t) - \theta_k(\cdot | x_t, a_t)\|_1 \right].$$

Let $\bar{\theta}_k(\cdot | x, a)$ be the empirical transition probability before the k th episode. We define \mathcal{M}_k as the set of models such that its transition probability θ satisfies $\|\bar{\theta}_k(\cdot | x, a) - \theta(\cdot | x, a)\|_1 \leq C \sqrt{\frac{S \ln(SAT)}{1 \vee N_{t_k}(x, a)}}$ for all $(x, a) \in \mathcal{S} \times \mathcal{A}$ where C is a universal constant which will be defined later. According to Theorem 9, we know that there exists a constant $C > 0$ such that $\Pr(M_k \notin \mathcal{M}_k) \leq 1/T$ and $\Pr(M^* \notin \mathcal{M}_k) \leq 1/T$.

Plugging in events $M_k \in \mathcal{M}_k, M^* \in \mathcal{M}_k$, we get

$$\begin{aligned} (III) &\leq D' \mathbb{E} \left[\sum_{k=1}^K \sum_{t=t_k}^{t_{k+1}-1} \|\theta^*(\cdot | x_t, a_t) - \theta_k(\cdot | x_t, a_t)\|_1 \cdot \mathbb{1}(M_k \in \mathcal{M}_k, M^* \in \mathcal{M}_k) \right] \\ &\quad + D' \mathbb{E} \left[\sum_{k=1}^K \sum_{t=t_k}^{t_{k+1}-1} \mathbb{1}(M_k \notin \mathcal{M}_k, M^* \notin \mathcal{M}_k) \right] \\ &\leq D' \mathbb{E} \left[\underbrace{\sum_{k=1}^K \sum_{t=t_k}^{t_{k+1}-1} C \sqrt{\frac{S \ln(SAT)}{1 \vee N_{t_k}(x_t, a_t)}}}_{(*)} + D' \mathbb{E} \left[\underbrace{\sum_{k=1}^K \sum_{t=t_k}^{t_{k+1}-1} \mathbb{1}(M_k \notin \mathcal{M}_k, M^* \notin \mathcal{M}_k)}_{(**)} \right] \right]. \quad (3) \end{aligned}$$

Note that for any $t_k \leq t < t_{k+1}$, we have $N_t(x, a) \leq 2N_{t_k}(x, a)$ holds for any state-action pair (x, a) . Hence $(t_{k+1} - t_k) \sqrt{\frac{1}{1 \vee N_{t_k}(s, a)}} \leq 2 \cdot \sum_{t=N_{t_k}(x, a)}^{N_{t_{k+1}-1}(x, a)} \sqrt{\frac{1}{1 \vee t}}$. Using this inequality in (*), we have

$$\begin{aligned} (*) &\leq \sum_{k=1}^K \sum_{t=t_k}^{t_{k+1}-1} C \sqrt{\frac{S \ln(SAT)}{1 \vee N_{t_k}(x_t, a_t)}} \\ &\leq 2C \cdot \sum_{k=1}^K \sum_{t=t_k}^{t_{k+1}-1} \sqrt{\frac{S \ln(SAT)}{1 \vee N_t(x_t, a_t)}} \\ &= 2C \cdot \sum_{t=1}^T \sqrt{\frac{S \ln(SAT)}{1 \vee N_t(x_t, a_t)}} \\ &= 2C \sqrt{S \ln(SAT)} \cdot \sum_{(x, a)} \sum_{t=0}^{N_T(x, a)} \sqrt{\frac{1}{1 \vee t}}. \end{aligned}$$

Since $\sum_{t=0}^{t'} \sqrt{\frac{1}{1+t}} \leq 2\sqrt{t'} + 1$, we further derive $(*) \leq 2C\sqrt{S \ln(SAT)} \cdot (SA + \sum_{(x,a)} \sqrt{N_T(x,a)}) \leq 2C\sqrt{S \ln(SAT)} \cdot (SA + \sqrt{SAT}) = \mathcal{O}(S\sqrt{AT \ln(SAT)})$, where the second last inequality is due to Cauchy-Schwarz inequality and the last inequality is due to $T \geq \sqrt{SA}$.

Recall that $\Pr(M_k \notin \mathcal{M}_k) \leq 1/T$ and $\Pr(M^* \notin \mathcal{M}_k) \leq 1/T$. Hence we have

$$\begin{aligned}
(**) &= \mathbb{E} \left[\sum_{k=1}^K \sum_{t=t_k}^{t_{k+1}-1} \mathbb{E}[\mathbf{1}(M_k \notin \mathcal{M}_k, M^* \notin \mathcal{M}_k)] \right] \\
&\leq \mathbb{E} \left[\sum_{k=1}^K \sum_{t=t_k}^{t_{k+1}-1} (\Pr(M_k \notin \mathcal{M}_k) + \Pr(M^* \notin \mathcal{M}_k)) \right] \\
&\leq 2.
\end{aligned} \tag{4}$$

Plugging in inequality $(*) \leq \mathcal{O}(S\sqrt{AT \ln(SAT)})$ and (4) back to (3), we get

$$(III) \leq \mathcal{O}(D'A\sqrt{AT \ln(SAT)}). \tag{5}$$

□

Putting Lemma 6, Lemma 7, Lemma 8 and (5) together, we prove this theorem.

5 Tools

Theorem 9 ([4]). *Let P be a probability distribution on the set $\mathcal{S} = \{1, \dots, S\}$. Let X_1, X_2, \dots, X_m be i.i.d. random variables distributed according to P . Then, for all $\epsilon > 0$, it holds that*

$$\Pr(\|P - \bar{P}\|_1 \geq \epsilon) \leq (2^S - 2) \exp(-m\epsilon^2/2),$$

where \bar{P} is the empirical estimation of P defined as $\bar{P}(i) = \frac{\sum_{j=1}^m \mathbf{1}(X_j=i)}{m}$.

References

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